Quantum minority game utilizing various forms of entanglement

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ABSTRACT

The quantum Minority game provides a means of studying the effect of multi-partite entanglement in a game theoretic setting. We study symmetric Nash equilibria and symmetric Pareto optimal strategies arising in a four-player quantum Minority game that uses an initial state that is a superposition of a GHZ state and products of EPR pairs. We find that the payoff curve for the symmetric Pareto optimal strategy is the same as that for the maximal violation of the Mermin-Ardehali-Belinskii-Klyshko inequality for the initial state, indicating a correspondence between quantum game theory and Bell inequalities. We also show that no advantage over the classical Minority game can be obtained when the initial state has only two party entanglement.

Keywords: Quantum games; Four-photon entanglement; Minority game; Bell inequalities

1. INTRODUCTION

The Minority game [1] is a simple multi-player game for studying strategic decision making within a group of agents. It arose as a model of repeated buying and selling in a stock market [2–4]. In each play, the agents independently select between one of two options, labeled '0' and '1' ('buy' and 'sell'). Those that choose the minority win and are awarded a payoff of one unit, while the others loose and receive no payoff. Players utilize knowledge of past successful choices to optimize their strategy. Examples of Minority game-like situations abound in everyday life. For example choosing which of two routes to use to drive into a city: the fastest route is the one that fewer other people have chosen.

Quantum versions of the Minority game have generated some interest [5–8]. They provide a means of studying multi-partite entanglement in a competitive setting using the tools of game theory, and for small numbers of players are amenable to simulation using multi-photon entangled states. Previous publications on the quantum Minority game (QMG) have considered only an initial GHZ state. In this paper we consider a four player QMG with various other forms of entanglement. Essentially the QMG can be considered as an optimization problem utilizing a multi-partite entangled state to produce results superior to those that can be achieved classically.

Our quantization of the Minority game is described as follows. Each of four players is given one qubit from a known four-partite entangled state. The players are permitted to act on their qubit with any local unitary operation. The choice of such an operator is the player's strategy. During this stage coherence is maintained and no communication between the players is permitted. After the player actions, a referee measures the qubits in the computational basis and awards payoffs using the classical payoff scheme.

When the initial state is a GHZ state our scheme is equivalent to the protocol of Eisert *et al.* [9], since the unentangling gate in that scheme has no effect on the payoffs for the Minority game [5]. Our scheme is also consistent with the generalized quantum game formalisms of Lee and Johnson [10] and Gutoski and Watrous [11]. It is represented schematically in Fig. 1.

The value of interest is the expectation value of the payoffs. In keeping with the game theoretic setting, the players are assumed to be rational independent agents that seek to maximize their payoff regardless of the actions of the other players. The generally accepted solution to a game is the Nash equilibrium (NE) [12], a strategy profile from which no player can improve their expected result by a unilateral change in strategy. Also

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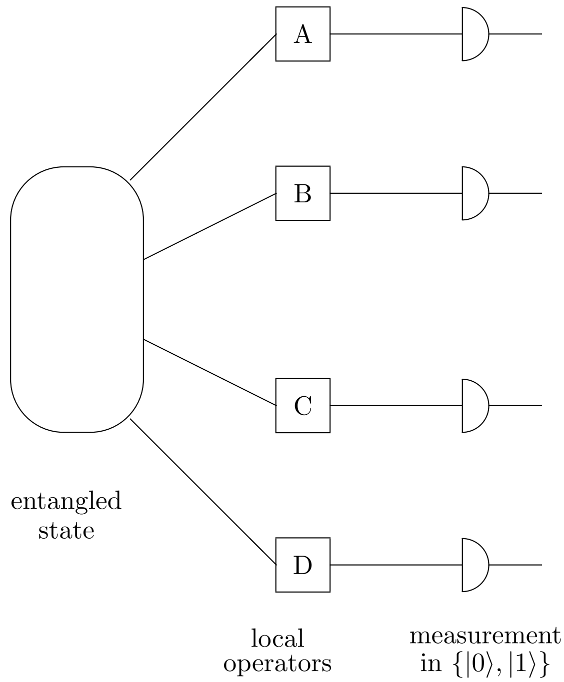

Figure 1. Schematic of a four player quantum Minority game.

of interest is the Pareto optimal (PO) equilibrium from which no player can improve their result without another being worse off.

The strategic space from which players can draw their strategy is SU(2). A convenient way of parameterizing these operators for our purposes is

$$\hat{M}(\theta, \beta_1, \beta_2) = \begin{pmatrix} e^{i\beta_1}\cos(\theta/2) & ie^{i\beta_2}\sin(\theta/2) \\ ie^{-i\beta_2}\sin(\theta/2) & e^{-i\beta_1}\cos(\theta/2) \end{pmatrix}, \tag{1}$$

where $\theta \in [0, \pi]$, $\beta_1, \beta_2 \in [-\pi, \pi]$. As shall become apparent later, for the QMG it generally suffices to use a restricted set where $\beta_1 = -\beta_2$.

With a view towards simulating the game using polarization encoded multi-partite entangled photon states generated by down-conversion [13–16] we shall take into account some loss of fidelity in the initial state but will consider the remainder of the process to be free from error. The QMG is particularly amenable to be played using post-selected entangled photon states since the objection that the entanglement is only known after a measurement has been taken and a four photon coincidence detected is not relevant. One can simply apply the appropriate operators to each photon channel, take a polarization measurement on each photon and only record those results for which a four photon coincidence was detected. Such detection indicates that the desired entangled state was present initially. Preliminary results for such an experiment have been obtained and will be presented elsewhere [17].

We are interested in symmetric equilibria, one where all players use the same strategy, since asymmetric equilibria are problematic to achieve in practice. In the classical case, such an equilibrium is provided by the mixed strategy of selecting each of '0' or '1' one half of the time. The resulting average payoff to each player in a four player Minority game is only $\frac{1}{8}$ since on half the occasions there is no minority.

The paper is arranged as follows. Sections 2 and 3 detail the calculations for a four player quantum Minority game with an initial state that is a superposition of a GHZ state and EPR pairs, showing that with these initial states any admixture of the GHZ state will produce a symmetric Pareto optimal payoff superior to that of the classical game. Sections 4 and 5 demonstrate that a quantum Minority game using only EPR pairs to entangle

nearest neighbors or all pairs of players does not give rise to any symmetric equilibria that have payoffs better than that achievable classically.

2. INITIAL STATE $\alpha|\mathrm{GHZ}\rangle + \sqrt{1-\alpha^2}|\mathrm{EPR}\rangle \otimes |\mathrm{EPR}\rangle$

The results for the four player QMG with an initial GHZ state were first determined by Benjamin and Hayden [5], and later generalized to multiple players [6] and to the consideration of decoherence [7]. We consider the initial state

$$|\psi_{\mathrm{in}}\rangle = \frac{\alpha}{\sqrt{2}}(|0000\rangle \ + \ |1111\rangle) \ + \ \frac{\sqrt{1-\alpha^2}}{2}\,(|01\rangle + |10\rangle) \otimes (|01\rangle + |10\rangle), \tag{2}$$

where $\alpha \in [0,1]$ is an adjustable parameter. Such states have been produced with polarization-encoded photons where α can be adjusted by rotating a polarizing beam splitter [17]. To make allowances for some loss of fidelity in the preparation of this state the initial state will be a mixed state represented by the density matrix

$$\rho_{\mathrm{in}} = p|\psi_{\mathrm{in}}\rangle\langle\psi_{\mathrm{in}}| + \frac{1-p}{16}\Sigma_{ijk\ell=0,1}\,|ijk\ell\rangle\langle ijk\ell|, \tag{3}$$

where $p \in [0,1]$ is a measure of the fidelity of the initial state. The second term in Eq. (3) represents completely random noise [13]. The state prior to measurement is

$$\rho_{\mathrm{final}} = (\hat{A} \otimes \hat{B} \otimes \hat{C} \otimes \hat{D})\rho_{\mathrm{in}}(\hat{A} \otimes \hat{B} \otimes \hat{C} \otimes \hat{D})^{\dagger}, \tag{4}$$

where $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are the operators chosen by the four players.

In the case where $\alpha = 1$ it is known that a symmetric NE occurs when all players choose the strategy [5]

$$\hat{M}(\pi/2, \pi/8, -\pi/8) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi/8} & ie^{-i\pi/8} \\ ie^{i\pi/8} & e^{-i\pi/8} \end{pmatrix}. \tag{5}$$

Although the value $\beta_1 = -\beta_2 = \pi/8$ is not the unique optimum, it is a focal point [18] that attracts the attention of the players since it is the simplest optimum value, and therefore there is no great difficulty in arriving at this NE. Given that at most one player of the four can be in the minority, $\frac{1}{4}$ is the greatest average payoff that can be expected. This is realized with the above strategy when the initial state has maximum fidelity. As $p \to 0$, the payoff reduces to that obtainable in a classical Minority game when the players all make a random selection. This is as expected since in the absence of entanglement the QMG cannot give any advantage over its classical counter part.

For general α, when three of the players use Eq. (5) while the fourth uses an arbitrary strategy, the payoff to the fourth player (Debra) is

$$\begin{aligned}\langle \$_{\mathrm{D}}\rangle = \frac{2-\alpha^2}{8} \ + \ \frac{\sin\theta}{16}\,\Big[&(3\sqrt{2}\alpha^2 \ - \ 2\sqrt{2} \ + \ 4\alpha\sqrt{1-\alpha^2})\cos(\beta_1-\beta_2) \\ &+ (3\sqrt{2}\alpha^2 \ - \ 2\sqrt{2} \ - \ 4\alpha\sqrt{1-\alpha^2})\sin(\beta_1-\beta_2)\Big],\end{aligned} \tag{6}$$

where we have set the fidelity p equal to one for simplicity. Since this, and other expressions for the expected payoff, depend only on the difference between the phases β_1 and β_2 we shall, without loss of generality, set $\beta_1 = -\beta_2 = \beta$. Clearly $\theta = \pi/2$ is optimal for the fourth player for all α, however, the value of β that maximizes Eq. (6) is dependent on α. The term in Eq. (6) in square brackets can be written as $A\sin(2\beta + \phi)$ where

$$\begin{aligned}\sin[\phi(\alpha)] &= \frac{3\sqrt{2}\alpha^2 \ - \ 2\sqrt{2} \ + \ 4\alpha\sqrt{1-\alpha^2}}{A}, \\ A &= 2(2-\alpha^2).\end{aligned} \tag{7}$$

The optimum value of 2β is thus $\pi/2 - \phi$, which gives

$$\begin{aligned}\langle \$_{\mathrm{A}}\rangle &= \frac{\alpha^4}{8-4\alpha^2}, \\ \langle \$_{\mathrm{D}}\rangle &= \frac{2-\alpha^2}{4},\end{aligned} \tag{8}$$

where the first line applies to any of the first three players, Alice, Bob or Charles. For $\alpha = 1$ the payoffs to all the players is $\frac{1}{4}$, as noted above. In this case there is no way that Debra, and by symmetry any of the other players, can improve their payoff by a unilateral change in strategy. For $\alpha < 1$, Debra can improve her payoff by switching from $\beta = \pi/8$ to $\beta = \pi/4 - \phi/2$. That is, Eq. (5) is a symmetric NE strategy for $\alpha = 1$ but not for other values of α.

In general, if there exists θ^*, β^* such that

$$\left\langle \$_{\rm D} \left(\hat{M}(\theta^*, \beta^*, -\beta^*)^{\otimes 4} \right) \right\rangle \geq \left\langle \$_{\rm D} \left(\hat{M}(\theta^*, \beta^*, -\beta^*)^{\otimes 3} \otimes \hat{M}(\theta, \beta, -\beta) \right) \right\rangle \quad \forall\, \theta, \beta, \tag{9}$$

then $\hat{M}(\theta^*, \beta^*, -\beta^*)$ is a symmetric NE strategy. There is no in principle objections to asymmetric NE strategy profiles, where the players choose different strategies, however in practice these cannot be reliably achieved in the absence of communication between the players since it is otherwise impossible for the players to know which of the different strategies to select. Necessary, but not sufficient, conditions for the existence of a symmetric NE are

$$\left. \frac{d\langle \$_{\rm D} \rangle}{d\theta} \right|_{\theta=\theta^*,\, \beta=\beta^*} = 0 \qquad \left. \frac{d\langle \$_{\rm D} \rangle}{d\beta} \right|_{\theta=\theta^*,\, \beta=\beta^*} = 0, \tag{10a}$$

$$\left. \frac{d^2\langle \$_{\rm D} \rangle}{d\theta^2} \right|_{\theta=\theta^*,\, \beta=\beta^*} \leq 0 \qquad \left. \frac{d^2\langle \$_{\rm D} \rangle}{d\beta^2} \right|_{\theta=\theta^*,\, \beta=\beta^*} \leq 0, \tag{10b}$$

where $\langle \$_{\rm D} \rangle$ is the payoff on the right hand side of Eq. (9). Inequalities in Eq. (10b) indicate a local maximum in the payoff to Debra, however this may not be a global maximum. An equality in Eq. (10b) may mean a local maximum, minimum or an inflection point in the payoff. We shall now enumerate all the symmetric NE strategies by considering Eqs. (10a–10b) over the range of $\alpha \in [0, 1]$.

If Alice, Bob and Charles use the strategy $\hat{M}(\theta, \beta, -\beta)$ while Debra plays $\hat{M}(\theta', \beta', -\beta')$ then

$$\begin{aligned} \left. \frac{d\langle \$_{\rm D} \rangle}{d\theta'} \right|_{\theta'=\theta,\, \beta'=\beta} &= \frac{\sin 2\theta}{8} \Big[2\alpha^2 \,+\, 2\alpha\sqrt{2-2\alpha^2}\, \cos 4\beta \\ &\qquad + (2\alpha^2 - 2 - 2\alpha\sqrt{2-2\alpha^2}\, \cos 4\beta - \alpha^2 \cos^2 4\beta) \sin^2\theta \Big] \\ \left. \frac{d\langle \$_{\rm D} \rangle}{d\beta'} \right|_{\theta'=\theta,\, \beta'=\beta} &= \frac{\alpha}{2} \sin 4\beta\, \sin^2\theta\, \left[(\sqrt{2-2\alpha^2} + \alpha \cos 4\beta) \sin^2\theta \,-\, 2\sqrt{2-2\alpha^2} \right]. \end{aligned} \tag{11}$$

We want to find θ, β for which these derivatives are simultaneously zero. Apart from the known NE for $\alpha = 1$ we find that the only other symmetric NE occurs for $\alpha \leq \sqrt{\frac{2}{3}}$ when $\cos 4\beta = 1$ and

$$\cos\theta = \sqrt{\frac{2 - 3\alpha^2}{2 - 2\alpha^2 + 2\alpha\sqrt{2-2\alpha^2}}}. \tag{12}$$

The expected payoff to each player for this equilibrium is

$$\langle \$ \rangle = \frac{\alpha(2-3\alpha^2)(\alpha + \sqrt{2-2\alpha^2})}{4 - 2\alpha^2 + 4\alpha\sqrt{2-2\alpha^2}}. \tag{13}$$

which reaches a maximum value of $(3 + 2\sqrt{3})/(18 + 10\sqrt{3}) \approx 0.183$ at $\alpha = \sqrt{\frac{1}{6}(3 - \sqrt{3})}$. Figs. 2 and 3 give the value of θ and the resulting payoff, respectively, for this solution.

Now consider the PO strategy amongst symmetric strategy profiles. That is, the choice of θ^*, β^* for which

$$\left\langle \$ \left(\hat{M}(\theta^*, \beta^*, -\beta^*)^{\otimes 4} \right) \right\rangle \geq \left\langle \$ \left(\hat{M}(\theta, \beta, -\beta)^{\otimes 4} \right) \right\rangle \quad \forall\, \theta, \beta, \tag{14}$$

where \$ represents the payoff to any one of the four players for the indicated strategy profile. Such a choice would yield the maximum possible payoff where all the players are restricted to the same strategy. Suppose all players select the strategy $\hat{M}(\theta, \beta, -\beta)$ for some θ, β to be determined. The payoff to each player is

$$\begin{aligned}\langle \$ \rangle = \frac{\sin^2\theta}{32}[&8 - 2\alpha^2 + 8\alpha\sqrt{2-2\alpha^2}\,\cos 4\beta - 2\alpha^2\cos 8\beta + 2(4-3\alpha^2)\cos 2\theta \\ &+ 8\alpha\sqrt{2-2\alpha^2}\,\cos 4\beta\,\cos 2\theta + 2\alpha^2\cos 8\beta\,\cos 2\theta\,].\end{aligned} \tag{15}$$

A local maximum or minimum in the value of the payoff will have $d\langle \$ \rangle/d\theta = d\langle \$ \rangle/d\beta = 0$ where,

$$\begin{aligned}\frac{d\langle \$ \rangle}{d\theta} &= \frac{\sin 2\theta}{4}[\alpha^2\sin 2\theta\,(1 - 2\cos^2 4\beta\,\sin^2\theta) \\ &\quad + \cos 2\theta\,(2 - 2\alpha^2 + 2\alpha\sqrt{2-2\alpha^2}\,\cos 4\beta)]\,; \end{aligned} \tag{16a}$$

$$\frac{d\langle \$ \rangle}{d\beta} = 2\alpha\sin^2\theta\,\sin 4\beta\,[\alpha\cos 4\beta\,\sin^2\theta - \sqrt{2-2\alpha^2}\,\cos^2\theta]. \tag{16b}$$

The latter expression is zero if $\sin\theta = 0$, $\sin 4\beta = 0$, or

$$\cos 4\beta = \frac{\sqrt{2-2\alpha^2}}{\alpha}\cot^2\theta. \tag{17}$$

Substituting the last expression into Eq. (16a) and simplifying gives

$$\frac{d\langle \$ \rangle}{d\theta} = \frac{3\alpha^2 - 2}{4}\sin 2\theta, \tag{18}$$

which is equal to zero if $\sin 2\theta = 0$ or $\alpha = \sqrt{\frac{2}{3}}$. From these results, Table 1 of extrema points and their corresponding payoffs can be calculated, where allowance has been made for non-unit fidelities p. Fig. 3 shows the payoffs for the first two strategies for a fidelity of $p = 1$, along with the NE payoff of Eq. (13). Note that the optimal strategy switches from $\hat{M}(\pi/2, \pi/8, -\pi/8)$ for $\alpha > \sqrt{\frac{2}{3}}$ where the initial state is dominated by the GHZ part, to $\hat{M}(\pi/4, 0, 0)$ for $\alpha < \sqrt{\frac{2}{3}}$, where the initial state is dominated by the EPR pairs. At $\alpha = \sqrt{\frac{2}{3}}$ the coefficients in the initial state, when expressed in the computational basis, are all equal and both strategies yield equivalent results.

There has been some recent interest in the correspondence between equilibria in quantum game theory and the violation of Bell inequalities [19, 20]. In our case we note that the curve for the symmetric PO payoff is the same as that for the violation of the Mermin-Ardehali-Belinski-Klyshko (MABK) inequality [21] shown in Fig. 4. This relationship will be investigated further in future work [22].

3. INITIAL STATE WITH SYMMETRIC EPR PAIRS

The initial state given in Eq. (2) is symmetric with respect to the interchange of any two qubits but not symmetric with respect to the interchange of any two pairs of qubits. It is interesting to consider the initial state that has this additional symmetry:

$$|\psi_{\text{in}}\rangle = \alpha|\text{GHZ}\rangle\ +\ \sqrt{\frac{1-\alpha^2}{3}}\ (|\text{EPR}\rangle_{\text{AB}}\otimes|\text{EPR}\rangle_{\text{CD}} + |\text{EPR}\rangle_{\text{AC}}\otimes|\text{EPR}\rangle_{\text{BD}} + |\text{EPR}\rangle_{\text{AD}}\otimes|\text{EPR}\rangle_{\text{BC}})\,, \tag{19}$$

where $|\text{GHZ}\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$ and $|\text{EPR}\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. The analysis proceeds in the same manner as in the previous section and we find the same PO strategies for the EPR-dominated and the GHZ-dominated regions, with a cross over at $\alpha = \sqrt{\frac{3}{4}}$. The direct correspondence between the PO payoffs and the maximal violation of the MABK inequality remains. Fig. 5 summarize these findings. Note that in the EPR-dominated region the optimal payoff curve reaches a maximum payoff of $\frac{1}{4}$ for $\alpha = \frac{1}{2}$, the same payoff as for the pure GHZ state.

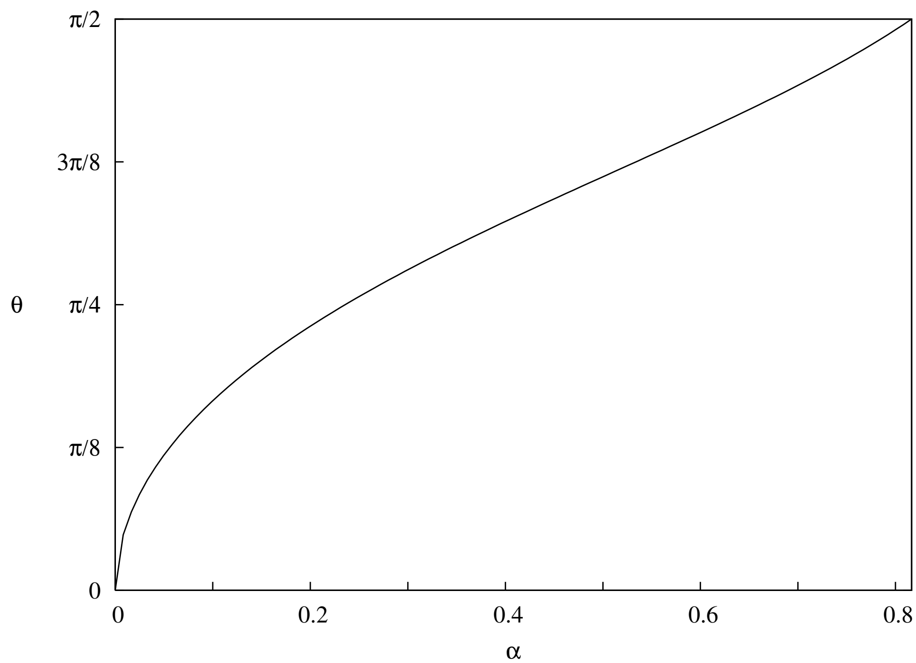

Figure 2. The value of θ given by Eq. (12) that, along with $\beta = 0$, gives a symmetric Nash equilibrium, as a function of the initial state parameter $\alpha \in [0, \sqrt{\frac{2}{3}}]$. The payoff for this equilibrium is given in Fig. 3 curve (c).

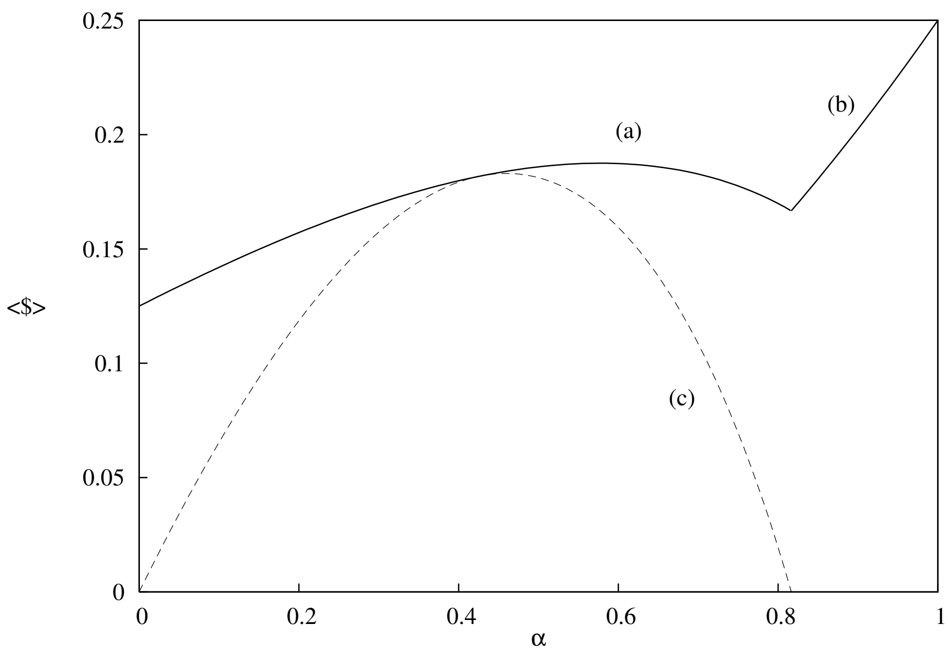

Figure 3. Payoffs for the two maxima in table 1, in (a) the EPR- and (b) the GHZ-dominated regions, along with (c) the symmetric Nash equilibrium (NE) payoff of Eq. (13), as a function of the initial state parameter α. The symmetric NE is maximal [i.e. (b) and (c) touch] when $\alpha = \sqrt{\frac{2}{11}}$, while the peak of (a) occurs when $\alpha = \sqrt{\frac{1}{3}}$. The two Pareto optimal payoff curves [(a) and (b)] meet at $\alpha = \sqrt{\frac{2}{3}}$.

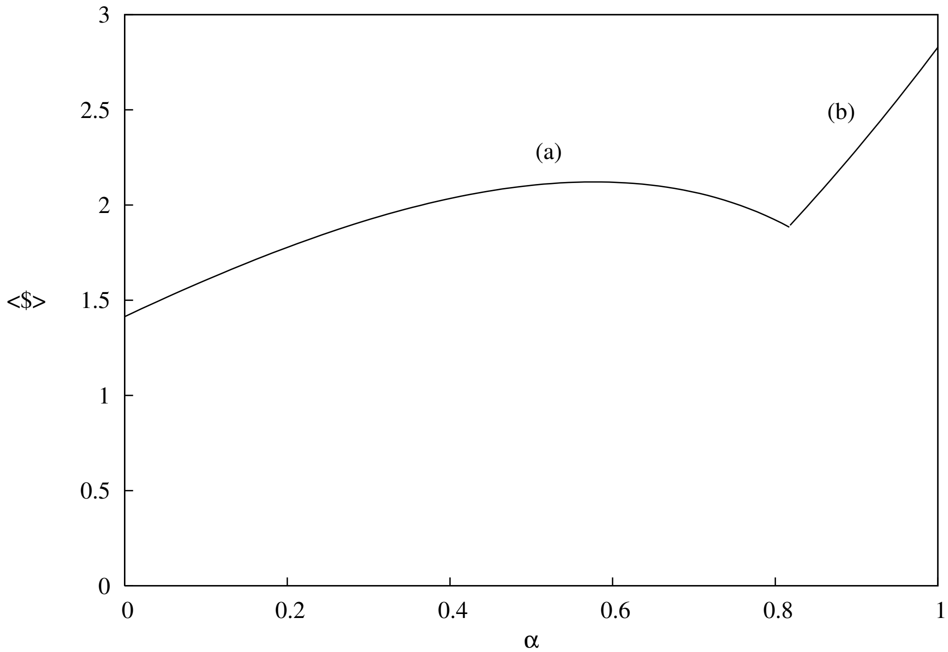

Figure 4. Violation of the MABK inequality for the initial state of Eq. (2) for (a) the EPR- and (b) GHZ-dominated regions. The curve is the same as the Pareto optimal payoff for the Minority game played with the same initial state given in Fig. 3 up to an arbitrary scaling factor. From Laskowski [23].

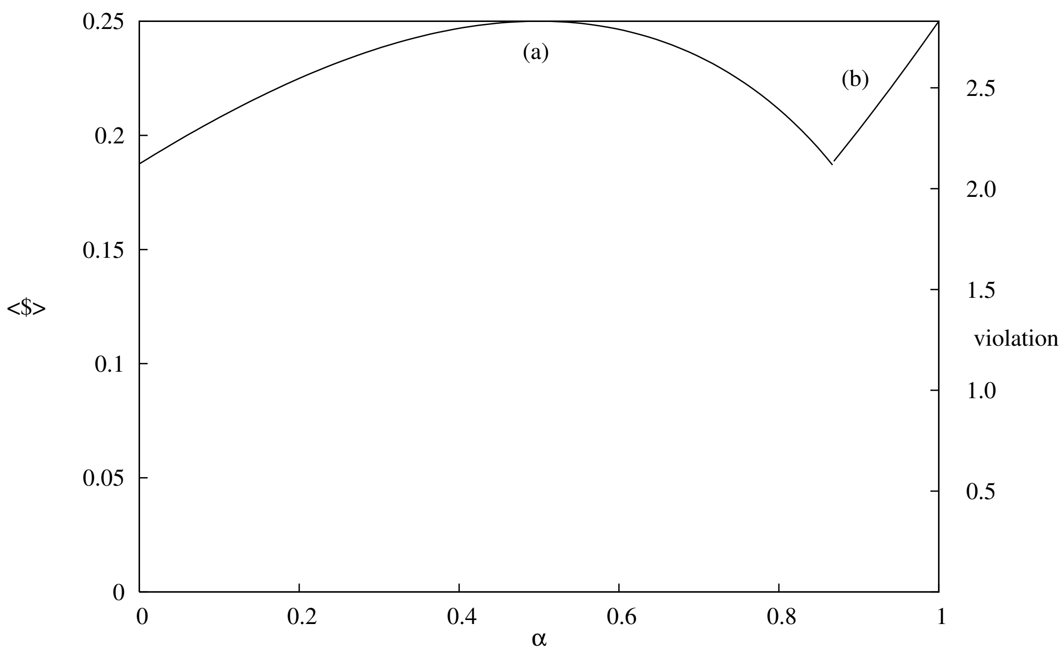

Figure 5. The Pareto optimal payoff curves for (a) the EPR- and (b) the GHZ-dominated regions for the quantum Minority game with initial state given by Eq. (19), as a function of the parameter α. The curves meet at $\alpha = \sqrt{\frac{3}{4}}$. The right hand scale gives the optimal violation of the MABK inequality for the initial state.

θ	β	$\langle\$\rangle$	$\frac{d^2\langle\$\rangle}{d\theta^2}$	$\frac{d^2\langle\$\rangle}{d\beta^2}$	
$\frac{\pi}{2}$	$\frac{\pi}{8}$	$\frac{1}{8}+\frac{p}{8}(2\alpha^2-1)$	<0 for $\alpha>\sqrt{\frac{2}{3}}$	<0	max for $\alpha>\sqrt{\frac{2}{3}}$
$\frac{\pi}{4}$	0	$\frac{1}{8}+\frac{p}{16}\,\alpha(2\sqrt{2-2\alpha^2}-\alpha)$	$<0\;\forall\,\alpha\neq\sqrt{\frac{2}{3}}$	<0 for $\alpha<\sqrt{\frac{2}{3}}$	max for $\alpha<\sqrt{\frac{2}{3}}$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{1}{8}+\frac{p}{16}\,\alpha(-2\sqrt{2-2\alpha^2}-\alpha)$	$<0\;\forall\,\alpha\neq\sqrt{\frac{2}{3}}$	>0	saddle
0	0	$\frac{1}{8}-\frac{p}{8}$	>0	0	min
0	$\frac{\pi}{4}$	$\frac{1}{8}-\frac{p}{8}$	>0	0	min
$\frac{\pi}{2}$	0	$\frac{1}{8}-\frac{p}{8}$	>0	>0	min
$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{1}{8}-\frac{p}{8}$	>0	>0	min

Table 1. Table of θ and β that give rise to a local maximum or minimum expected payoff for a symmetric strategy profile $\hat{M}(\theta,\beta,-\beta)^{\otimes 4}$, along with the resulting payoff. The second derivatives of the payoff can be used to classify the extrema. Here, $p\in[0,1]$ is the fidelity to which the initial state is prepared.

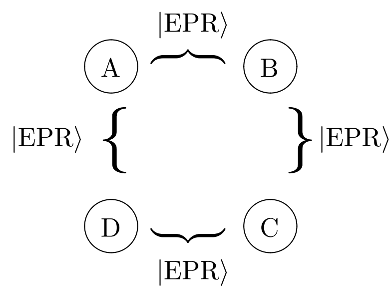

Figure 6. Schematic of the arrangement of the initial state in a four player quantum Minority game with nearest neighbor entanglement.

4. QUANTUM MINORITY GAME WITH NEAREST NEIGHBOR ENTANGLEMENT

Instead of playing the four player QMG with a four-partite entangled initial state we shall consider instead utilizing only two qubit entanglement. Such a scheme has been applied with success in a quantum version of the Public Goods game [24]. A quantum version of the Minority Game can be constructed using only two qubit EPR states as follows. Each player has two qubits. Initially, one is entangled with a qubit of the player to their right in an EPR pair, while the other is entangled with a qubit of the player to their left in an EPR pair. That is, the initial state for a four player game is the eight qubit state

$$
\begin{aligned}
|\psi_i\rangle &= |q_{\mathrm{R}}^{(\mathrm{D})}\, q_{\mathrm{L}}^{(\mathrm{A})}\, q_{\mathrm{R}}^{(\mathrm{A})}\, q_{\mathrm{L}}^{(\mathrm{B})}\, q_{\mathrm{R}}^{(\mathrm{B})}\, q_{\mathrm{L}}^{(\mathrm{C})}\, q_{\mathrm{R}}^{(\mathrm{C})}\, q_{\mathrm{L}}^{(\mathrm{D})}\rangle \\
&= \frac{1}{4}(|00\rangle + i|11\rangle)^{\otimes 4},
\end{aligned} \tag{20}
$$

where $q^{(k)}$ is a qubit of player k, and subscripts L and R refer to the qubit entangled with the player to the left or right, respectively. The strategy of a player consists of two single qubit unitary operators, one acting on their left qubit and the other on their right qubit. Again we will restrict ourselves to operators where $\beta_1=-\beta_2$ but one can show with a full analysis that this can be done without loss of generality since only the difference $\beta_1-\beta_2$ is relevant. After all the players have applied their operators each player measures their two qubits in the computational basis and selects as their move a logical one if the parity of the (measured) bits is odd or a logical zero if the parity of the bits is even. Payoffs can then be assigned as in the classical game. The initial state is indicated schematically in Fig. 6.

We look for symmetric NE in a manner analogous to that of Eq. (9). That is, the strategy $(\hat{M}_{\mathrm{L}}^*,\hat{M}_{\mathrm{R}}^*)$ is a

symmetric NE strategy iff

$$\left\langle \$_{\rm D}\left(\hat{M}_{\rm R}^{*}\otimes(\hat{M}_{\rm L}^{*}\otimes\hat{M}_{\rm R}^{*})^{\otimes 3}\otimes\hat{M}_{\rm L}^{*}\right)\right\rangle \geq \left\langle \$_{\rm D}\left(\hat{M}_{\rm R}\otimes(\hat{M}_{\rm L}^{*}\otimes\hat{M}_{\rm R}^{*})^{\otimes 3}\otimes\hat{M}_{\rm L}\right)\right\rangle \quad \forall\, \hat{M}_{\rm L},\hat{M}_{\rm R}, \tag{21}$$

where $\hat{M}_{\rm L}^{*} \equiv \hat{M}(\theta_{\rm L}^{*},\beta_{\rm L}^{*},-\beta_{\rm L}^{*})$ and $\hat{M}_{\rm L} \equiv \hat{M}(\theta_{\rm L},\beta_{\rm L},-\beta_{\rm L})$, and similarly for $\hat{M}_{\rm R}^{*},\hat{M}_{\rm R}$. Below I shall use the shorthand $(\theta_{\rm L},\beta_{\rm L},\theta_{\rm R},\beta_{\rm R})$ to mean the strategy $\hat{M}(\theta_{\rm L},\beta_{\rm L},-\beta_{\rm L})$ applied to the left qubit and $\hat{M}(\theta_{\rm R},\beta_{\rm R},-\beta_{\rm R})$ to the right.

Suppose Debra plays $(\theta'_{\rm L},\beta'_{\rm L},\theta'_{\rm R},\beta'_{\rm R})$ while the others play $(\theta_{\rm L},\beta_{\rm L},\theta_{\rm R},\beta_{\rm R})$. The full expression for Debra's payoff $\langle\$\rangle$ is too long to reproduce here. We are only interested in the derivatives of the payoff with respect to Debra's parameters:

$$\begin{aligned}\frac{d\langle\$\rangle}{d\theta_{\rm L}} &= \frac{1}{8}[\cos\theta_{\rm R}\,\sin\theta_{\rm L} \,+\, \cos\theta_{\rm L}\,\sin\theta_{\rm R}\sin(2\beta_{\rm L}+2\beta_{\rm R})]\\ &\quad\times[\cos\theta_{\rm L}\,\cos\theta_{\rm R} \,-\, \sin\theta_{\rm L}\,\sin\theta_{\rm R}\,\sin(2\beta_{\rm L}+2\beta_{\rm R})]^3,\end{aligned} \tag{22a}$$

$$\begin{aligned}\frac{d\langle\$\rangle}{d\theta_{\rm R}} &= \frac{1}{8}[\cos\theta_{\rm L}\,\sin\theta_{\rm R} \,+\, \cos\theta_{\rm R}\,\sin\theta_{\rm L}\sin(2\beta_{\rm L}-2\beta_{\rm R})]\\ &\quad\times[\cos\theta_{\rm L}\,\cos\theta_{\rm R} \,-\, \sin\theta_{\rm L}\,\sin\theta_{\rm R}\,\sin(2\beta_{\rm L}+2\beta_{\rm R})]^3,\end{aligned} \tag{22b}$$

$$\begin{aligned}\frac{d\langle\$\rangle}{d\beta_{\rm L}} &= \frac{d\langle\$\rangle}{d\beta_{\rm R}}\\ &= \frac{1}{4}\sin\theta_{\rm L}\,\sin\theta_{\rm R}\,\cos(2\beta_{\rm L}+2\beta_{\rm R})\\ &\quad\times[\cos\theta_{\rm L}\,\cos\theta_{\rm R} \,-\, \sin\theta_{\rm L}\,\sin\theta_{\rm R}\,\sin(2\beta_{\rm L}+2\beta_{\rm R})]^3,\end{aligned} \tag{22c}$$

where all derivative are evaluated at $\theta'_{\rm L}=\theta_{\rm L},\theta'_{\rm R}=\theta_{\rm R},\beta'_{\rm L}=\beta_{\rm L},\beta'_{\rm R}=\beta_{\rm R}$. Stationary points occur when both $\theta_{\rm L}$ and $\theta_{\rm R}$ are $0,\pi/2$ or π, or when

$$\cot\theta_{\rm L}\,\cot\theta_{\rm R} = \sin(2\beta_{\rm L}+2\beta_{\rm R}). \tag{23}$$

Table 2 gives the NE strategies along with their payoffs. As is clear from the table the best NE have an expected payoff of $\frac{1}{8}$, the same as that for the classical game. Thus, contrary to the example in Ref. [24], no advantage is obtained from merely having nearest neighbor (two party) entanglement.

$\theta_{\rm L}$	$\theta_{\rm R}$	$\beta_{\rm L}$	$\beta_{\rm R}$	$\langle\$\rangle$
0	0	–	–	0
0	$\frac{\pi}{2}$	–	–	$\frac{1}{8}$
$\frac{\pi}{2}$	0	–	–	$\frac{1}{8}$
$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\beta_{\rm L}$	$\beta_{\rm R}$	$\frac{1}{8}\cos^2(2\beta_{\rm L}+2\beta_{\rm R})(2-\cos^2(2\beta_{\rm L}+2\beta_{\rm R}))$
$\cot\theta_{\rm L}\,\cot\theta_{\rm R}=\sin(2\beta_{\rm L}+2\beta_{\rm R})$				$\frac{1}{8}$

Table 2. Table of symmetric Nash equilibria for a four player quantum Minority game with nearest neighbor entanglement, where everyone plays $(\theta_{\rm L},\beta_{\rm L},\theta_{\rm R},\beta_{\rm R})$. In the first three lines $\theta=\pi$ is equivalent to $\theta=0$ and the value of the β's does not matter. The maximum payoff in the fourth line is $\frac{1}{8}$.

Now consider the optimal symmetric strategy. That is, the strategy profile that maximizes the expected payoff to each player, where everyone plays $(\theta_{\rm L},\theta_{\rm R},\beta_{\rm L},\beta_{\rm R})$. The derivatives of the expected payoff with respect to each of the parameters are the same as those given in Eqs. (22a–22c) up to a constant factor, and hence the stationary points for the expected payoff are just those given in Table 2. We conclude that the four player QMG with nearest neighbor (two party) entanglement offers no advantage to the players over the classical game.

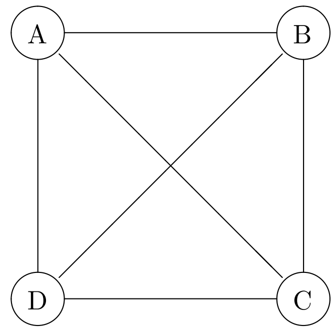

Figure 7. Schematic of the arrangement of the initial state in a four player quantum Minority game with two party entanglement between each pair. The lines represent shared EPR states.

5. QUANTUM MINORITY GAME WITH TWO-PARTY ENTANGLEMENT BETWEEN EACH PAIR

An extension of the version of the QMG considered in the previous section is to increase the number of EPR pairs shared between the players so that all pairs of players share some entanglement. Such a scheme has also been considered in the quantum Public Goods game [24]. In the four player QMG three qubits are required for each player. A schematic of the arrangement of the initial state is shown in Fig. 7.

The initial state can be written as

$$\begin{aligned} |\psi_i\rangle &= |q_{\mathrm{R}}^{(\mathrm{D})}\, q_{\mathrm{L}}^{(\mathrm{A})}\, q_{\mathrm{R}}^{(\mathrm{A})}\, q_{\mathrm{L}}^{(\mathrm{B})}\, q_{\mathrm{R}}^{(\mathrm{B})}\, q_{\mathrm{L}}^{(\mathrm{C})}\, q_{\mathrm{R}}^{(\mathrm{C})}\, q_{\mathrm{L}}^{(\mathrm{D})}\, q_{\mathrm{M}}^{(\mathrm{A})}\, q_{\mathrm{M}}^{(\mathrm{C})}\, q_{\mathrm{M}}^{(\mathrm{B})}\, q_{\mathrm{M}}^{(\mathrm{D})}\rangle \\ &= \frac{1}{8}(|00\rangle + i|11\rangle)^{\otimes 6}, \end{aligned} \tag{24}$$

where $q^{(k)}$ is a qubit of player k, and subscripts L,R and M refer to the qubit entangled with the player to the left, right or middle, respectively. The strategy of a player consists of three single qubit unitary operators, acting on their left, right and middle qubits. Again only the difference between the phases $\beta_1 - \beta_2$ is relevant so we can restrict ourselves to operators where $\beta_1 = -\beta_2$ without loss of generality. After all the players have applied their operators each player measures their three qubits in the computational basis and selects as their move a logical one or zero depending on the parity of their (measured) bits. To avoid a bias, Alice and Bob will choose a logical one (zero) if their parity is even (odd), while Charles and Debra will do the opposite. Again, payoffs can then be assigned as in the classical game.

Proceeding as in the previous section, except with an additional operator for each player we find that a necessary condition for a strategy to be symmetric NE strategy is either

$$\cot\theta_{\mathrm{L}}\, \cot\theta_{\mathrm{R}} = \sin(2\beta_{\mathrm{L}} + 2\beta_{\mathrm{R}}), \tag{25a}$$

$$\text{or} \qquad \sin 4\beta_{\mathrm{M}} = \frac{1+\cos 2\theta_{\mathrm{M}}}{1-\cos 2\theta_{\mathrm{M}}}. \tag{25b}$$

Given that any set of $(\theta_{\mathrm{L}}, \beta_{\mathrm{L}}, \theta_{\mathrm{M}}, \beta_{\mathrm{M}}, \theta_{\mathrm{R}}, \beta_{\mathrm{R}})$ that satisfy Eqs. (25a–25b) represents a stationary point for the payoff we are free to choose any values of the parameters that satisfy one or other of the conditions. In this way we are able to easily evaluate the payoff to the players in both cases, resulting in $\langle\$\rangle = \frac{1}{8}$. Hence, there is no NE in this set up that has a payoff greater than that of the classical game.

The optimal symmetric strategy profile can be calculated in the same way as in the previous section. Here we assume all players choose the same strategy $(\theta_{\mathrm{L}}, \beta_{\mathrm{L}}, \theta_{\mathrm{M}}, \beta_{\mathrm{M}}, \theta_{\mathrm{R}}, \beta_{\mathrm{R}})$. We then compute the derivatives of the payoff with respect to variations in this strategy and find the stationary points. The derivatives are the same as those for the symmetric NE calculations, up to a constant factor, and so the conditions for stationarity are Eqs. (25a–25b) as before, yielding the classical NE payoff of $\frac{1}{8}$.

6. CONCLUSION

Playing a four person quantum Minority game using an initial state that is a superposition of the GHZ state with pairs of EPR states yields an interesting quantum game, with a symmetric Pareto optimal payoff superior to that of the classical game for any non-zero admixture of the GHZ state. There are two regions, one dominated by the GHZ state and the other by the EPR pairs. The optimal strategy switches between regions with the "quantum fulcrum" occurring where the coefficients in the superposition, when expressed in the computational basis, are equal. The optimal symmetric payoff is the same as the violation of the MABK inequality for the initial state up to an (arbitrary) constant scaling factor. For the inequalities, the observables giving rise to the maximum violation switch at the same fulcrum as that for the optimal strategies. This correspondence warrants further investigation.

When the players of the quantum game have access only to EPR pairs to entangle the players, either to their nearest neighbors or to every other player, no advantage over the classical Minority game is obtained. This is in contrast to the situation for an N-party quantum Public Goods game, where two-party entanglement was as good as entanglement between all parties through a multi-partite GHZ state.

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