Multiplayer quantum Minority game with decoherence

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ABSTRACT

A quantum version of the Minority game for an arbitrary number of agents is studied. When the number of agents is odd, quantizing the game produces no advantage to the players, however, for an even number of agents new Nash equilibria appear that have no classical analogue. The new Nash equilibria provide far preferable expected payoffs to the players compared to the equivalent classical game. The effect on the Nash equilibrium payoff of reducing the degree of entanglement, or of introducing decoherence into the model, is indicated.

Keywords: Quantum games, decoherence, Minority game

1. INTRODUCTION

Game theory is the formal description of conflict or competition situations where the outcome is contingent upon the interaction of the strategies of the various agents. For every outcome, each player assigns a numerical measure of the desirability to them of that outcome, known as their utility or payoff.* A solution of a game-theoretic problem is a strategy profile that represents some form of equilibrium, the best known of which is the Nash equilibrium[1] (NE) from which no player can improve their payoff by a unilateral change in strategy. Originally developed for use in economics,[2] game theory is now a mature branch of mathematics used in the social and biological sciences, computing and, more recently, in the hard sciences.[3]

The Minority game, initially proposed by Challet and Zhang,[4] has received much attention as a model of a population of agents repeatedly competing for limited resources.[5–7] In its simplest form, at each step the agents must independently select among a pair of choices, labeled '0' and '1.' Players selecting the least popular choice are rewarded with a unit payoff while the majority emerge empty handed. Players' strategies can be based on knowledge of previous selections and successes in past rounds. The idea behind the Minority game is neatly encapsulated by the following quote:

> It is not worth an intelligent man's time to be in the majority. By definition there are already enough people to do that—*Geoffery Harold Hardy.*

The Minority game is generally restricted to an odd number of agents, but even numbers can be permitted with the proviso that when the number of players selecting 0 and 1 are equal all players score zero.

A game can be considered an information processing system, where the players' strategies are the input and the payoffs are the output. With the advent of quantum computing and the increasing interest in quantum information[8,9] it is natural to consider the combination of quantum mechanics and game theory. Papers by Meyer[10] and Eisert *et al.*[11] paved the way for the creation of the new field of quantum game theory. Classical probabilities are replaced by quantum amplitudes and players can utilize superposition, entanglement and interference.

In quantum game theory, new ideas arise in two-player[12–18] and multiplayer settings.[19–24] In the protocol of Eisert *et al.*,[11] in two player quantum games there is no NE when both players have access to the full set of unitary strategies.[25] Nash equilibria exist amongst mixed quantum[12] strategies† or when the strategy set is

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*Strictly, the utility is a numerical measure and the payoff is a relative ordering, but for the purpose of the present work the two terms shall be used interchangeably.

†Strategies are referred to as pure when the actions of the player at any stage is deterministic and mixed when a randomizing device is used to select among actions. That is, a mixed strategy is a convex linear combination of pure strategies.

restricted in some way.[11, 26, 27] However, in multiplayer quantum games new NE amongst unitary strategies can arise.[19] These new equilibria have no classical analogue. Reviews of quantum games and their applications are given by Flitney and Abbott[28] and Piotrowski and Sładowski.[29, 30]

The realization of quantum computing is still an endeavour that faces great challenges.[31] A major hurdle is the maintainance of coherence during the computation, without which the special features of quantum computation are lost. Decoherence results from the coupling of the system with the environment and produces non-unitary dynamics. Interaction with the environment can never be entirely eliminated in any realistic quantum computer. Zurek gives a review of the standard mechanisms of quantum decoherence.[32] By encoding the logical qubits in a number of physical qubits, quantum computing in the presence of noise is possible. Quantum error correcting codes[33] function provided the error rate is small enough, while decoherence free subspaces[34] eliminate certain types of decoherence. The disadvantage of both techniques is the expansion of the number of qubits required for a calculation.

The theory of quantum control in the presence of noise is little studied. Johnson has considered a three-player quantum game where the initial state is flipped to $|111\rangle$ from the usual $|000\rangle$ with some probability,[35] while Özdemir *et al.*[36] have considered various two-player, two strategy (2×2) quantum games where the initial state is corrupted by bit-flip errors. In both papers it was found that quantum effects impede the players above a certain level of noise. They are then better off playing the classical game. Chen *et al.* found that the NE in a set of restricted quantum strategies was unaffected by decoherence in quantum Prisoners' Dilemma.[37] Decoherence in various two player quantum games in the Eisert protocol is considered by Flitney and Abbott.[38, 39] The quantum player maintains an advantage over a player restricted to classical strategies provided some level of coherence remains. The current work considers a quantum version of the Minority game for an arbitrary number of agents in the presence of decoherence.

2. QUANTUM GAMES WITH DECOHERENCE

The standard protocol for quantizing a game is well described in a number of papers[11, 28, 40] and will be covered here only briefly. If an agent has a choice between two strategies, the selection can be encoded in the classical case by a bit. To translate this into the quantum realm the bit is altered to a qubit, with the computational basis states $|0\rangle$ and $|1\rangle$ representing the original classical strategies. The initial game state consists of one qubit for each player, prepared in a maximally entangled state by an entangling operator $\hat{J}$ acting on $|00\ldots0\rangle$. Pure quantum strategies are local unitary operators acting on a player's qubit. After all players have executed their moves the game state undergoes a positive operator valued measurement and the payoffs are determined from the classical payoff matrix. In the Eisert protocol this is achieved by applying $\hat{J}^\dagger$ to the game state and then making a measurement in the computational basis state. That is, the state prior to the measurement in the N-player case can be computed by

$$|\psi_f\rangle = \hat{J}^\dagger (\hat{M}_1 \otimes \hat{M}_2 \otimes \ldots \otimes \hat{M}_N)\, \hat{J} |\psi_0\rangle \;, \tag{1}$$

where $|\psi_0\rangle = |00\ldots0\rangle$ represents the initial state of the N qubits, $\hat{J}$ is an operator that entangles the players' qubits, and $\hat{M}_k,\ k = 1, \ldots, N$, is a unitary operator representing the move of player k. The classical pure strategies are represented by the identity and the bit-flip operator. The entangling operator $\hat{J}$ commutes with any direct product of classical moves, so the classical game is simply reproduced if all players select a classical move.

To consider decoherence it is most convenient to use the density matrix notation for the state of the system and the operator sum representation for the quantum operators, despite the well known limitations of this representation.[41] Lagrangian field theory, path integrals, master equations, quantum Langevin equations, short-time perturbation expansions, Monte-Carlo methods, semiclassical and phenomenological methods are some of the other techniques for calculating decoherence.[42] Decoherence includes dephasing, which randomizes the relative phase between the $|0\rangle$ and $|1\rangle$ states, and dissipation, that modifies the populations of the states, amongst other forms.[8] Pure dephasing can be expressed as

$$a|0\rangle \;+\; b|1\rangle \rightarrow a|0\rangle \;+\; b\, e^{i\phi}|1\rangle. \tag{2}$$

If the phase shift ϕ is a random variable with a Gaussian distribution of mean zero and variance 2λ, the density matrix obtained after averaging over all values of ϕ is[8]

$$\begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix} \rightarrow \begin{pmatrix} |a|^2 & a\bar{b}\,e^{-\lambda} \\ \bar{a}b\,e^{-\lambda} & |b|^2 \end{pmatrix}. \tag{3}$$

Thus, over time, dephasing causes an exponential decay of the off-diagonal elements of the density matrix.

Making a measurement with probability p in the $\{|0\rangle, |1\rangle\}$ basis on a qubit described by the density matrix ρ can be represented in the operator sum formalism by

$$\rho \;\rightarrow\; \sum_{j=0}^{2} \mathcal{E}_j\,\rho\,\mathcal{E}_j^\dagger, \tag{4}$$

where $\mathcal{E}_0 = \sqrt{p}\,|0\rangle\langle 0|$, $\mathcal{E}_1 = \sqrt{p}\,|1\rangle\langle 1|$, and $\mathcal{E}_2 = \sqrt{1-p}\,\hat{I}$. By the addition of further $\mathcal{E}_j$'s an extension to N qubits is achieved:

$$\rho \;\rightarrow\; \sum_{j_1,\ldots,j_N=0}^{2} \mathcal{E}_{j_1} \otimes \ldots \otimes \mathcal{E}_{j_N}\,\rho\,\mathcal{E}_{j_N}^\dagger \otimes \ldots \otimes \mathcal{E}_{j_1}^\dagger, \tag{5}$$

where, here, ρ is an n-qubit state. By identifying $1-p = e^{-\lambda}$, the measurement process as described has the same results as pure dephasing: the exponential decay of the off-diagonal elements of ρ.

A quantum game in the Eisert scheme with decoherence can be described in the following manner

$$\begin{aligned}
\rho_i \equiv \rho_0 &= |\psi_0\rangle\langle\psi_0| && \text{(initial state)} \\
\rho_1 &= \hat{J}\rho_0\hat{J}^\dagger && \text{(entanglement)} \\
\rho_2 &= D(\rho_1, p_1) && \text{(partial decoherence)} \\
\rho_3 &= (\otimes_{k=1}^{N}\hat{M}_k)\,\rho_2\,(\otimes_{k=1}^{N}\hat{M}_k)^\dagger && \text{(players' moves)} \\
\rho_4 &= D(\rho_3, p_2) && \text{(partial decoherence)} \\
\rho_5 &= \hat{J}^\dagger\rho_4\hat{J} && \text{(preparation for measurement),}
\end{aligned} \tag{6}$$

to produce the final state $\rho_f \equiv \rho_5$ upon which a measurement is taken. The function $D(\rho, p)$ is a completely positive map that applies some form of decoherence to the state ρ controlled by the probability p. The scheme is shown in Figure 1. The expectation value of the payoff to the kth player is

$$\langle \$^k \rangle = \sum_{\xi} \hat{\mathcal{P}}_\xi\,\rho_f\,\hat{\mathcal{P}}_\xi^\dagger\,\$^k_\xi, \tag{7}$$

where $\hat{\mathcal{P}}_\xi = |\xi\rangle\langle\xi|$ is the projector onto the state $|\xi\rangle$, $\$^k_\xi$ is the payoff to the kth player when the final state is $|\xi\rangle$, and the summation is taken over $\xi = j_1 j_2 \ldots j_N,\; j_i \in \{0, 1\}$.

3. RESULTS FOR THE MULTIPLAYER MINORITY GAME

This paper only considers the situation where players do not make use of their knowledge of past successes, but simply have the classical strategies "always choose 0" or "always choose 1." A pure quantum strategy is an SU(2) operator:

$$\hat{U}(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix}, \tag{8}$$

where $\theta \in \{0, \pi\}$ and $\alpha, \beta \in \{-\pi, \pi\}$. The kth player's move is $\hat{U}(\theta_k, \alpha_k, \beta_k)$. Here, $\hat{I} \equiv \hat{U}(0,0,0)$ and $\hat{F} \equiv \hat{U}(\pi, 0, 0)$ correspond to the two classical moves. Entanglement is achieved by

$$\hat{J} = \frac{1}{\sqrt{2}}(\hat{I}^{\otimes N} + i\sigma_x^{\otimes N}). \tag{9}$$

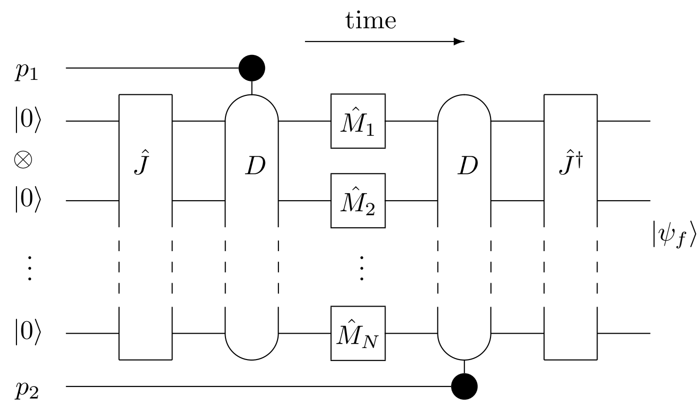

Figure 1. The flow of information in an N-person quantum game with decoherence, where M_k is the move of the kth player and $\hat{J}$ ($\hat{J}^\dagger$) is an entangling (dis-entangling) gate. The central horizontal lines are the players' qubits and the top and bottom lines are classical random bits with a probability p_1 or p_2, respectively, of being 1. Here, D is some form of decoherence controlled by the classical bits. Figure from Flitney and Abbott.[39]

Operators of the form $\hat{U}(\theta, 0, 0)$ are equivalent to classical mixed strategies, with the mixing controlled by θ, since when all players use these strategies the quantum game reduces to the classical one. There is some arbitrariness about the representation of the operators. Other representations may lead to a different overall phase in the final state, but this has no physical significance.

Benjamin and Hayden showed that in the four player quantum Minority game an optimal strategy arises:[19]

$$\begin{aligned} \hat{s}_{\rm NE} &= \frac{1}{\sqrt{2}}\cos(\frac{\pi}{16})(\hat{I} + i\hat{\sigma}_x) \; - \; \frac{1}{\sqrt{2}}\sin(\frac{\pi}{16})(i\hat{\sigma}_y + i\hat{\sigma}_z) \\ &= \hat{U}(\frac{\pi}{2}, \frac{-\pi}{16}, \frac{\pi}{16}). \end{aligned} \tag{10}$$

The strategy profile $\{\hat{s}_{\rm NE}, \hat{s}_{\rm NE}, \hat{s}_{\rm NE}, \hat{s}_{\rm NE}\}$ results in a NE with an expected payoff of $\frac{1}{4}$ to each player, the maximum possible from a symmetric strategy profile, and twice that that can be achieved in the classical game where the players can do no better than selecting 0 or 1 at random. The optimization is the result of the elimination of the states for which no player scores: those where all the players make the same selection or where the choices are balanced.

The strategy of Eq. (10) is seen to be a NE by observing the payoff to the first player when they vary from the NE profile by selecting the general strategy $\hat{U}(\theta, \alpha, \beta)$. Figure 2 shows the first player's payoff as a function of θ when $\beta = -\alpha = \pi/16$, and as a function of α and β when $\theta = \pi/2$. The latter figure indicates that the NE is not strict: varying the strategy to $\hat{U}(\pi/2, \eta - \pi/16, \eta + \pi/16)$, for arbitrary $\eta \in \{-15\pi/16, 15\pi/16\}$ leaves the payoff unchanged.

The addition of decoherence to the four player quantum Minority game results in a diminution of the NE payoff, ultimately to the classical value of $\frac{1}{8}$ when the decoherence probability p is maximized, as indicated in figure 3. However, the strategy given by Eq. (10) remains a NE for all $p < 1$. This is in contrast with the results of Johnson[35] and Özdemir *et al.*[36] who showed that the quantum optimization did not survive above a certain noise threshold in the quantum games they considered.

In an N-player quantum Minority game a symmetric NE profile can be found by considering

$$\hat{s}_\delta = \hat{U}(\frac{\pi}{2}, -\delta, \delta). \tag{11}$$

When all players choose this strategy, for even N the coefficient of states that have an equal number of ones and zeros is proportional to $\cos(N\delta) - \sin(N\delta)$, giving a probability for these states proportional to $1 - \sin(2N\delta)$. This probability vanishes when $\delta = (4n+1)\pi/(4N)$, $n = 0, \pm 1, \pm 2, \ldots$ For the collective good, the vanishing

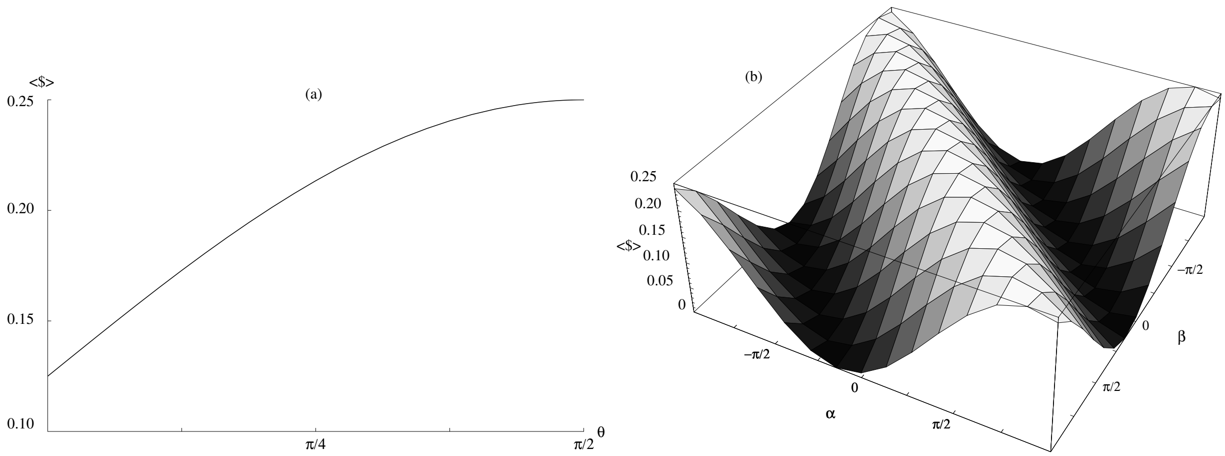

Figure 2. The payoff to the first player in a four player quantum Minority game when they choose the strategy (a) $\hat{U}(\theta, -\pi/16, \pi/16)$ or (b) $\hat{U}(\pi/2, \alpha, \beta)$, while the other players all select $\hat{s}_{\rm NE}$.

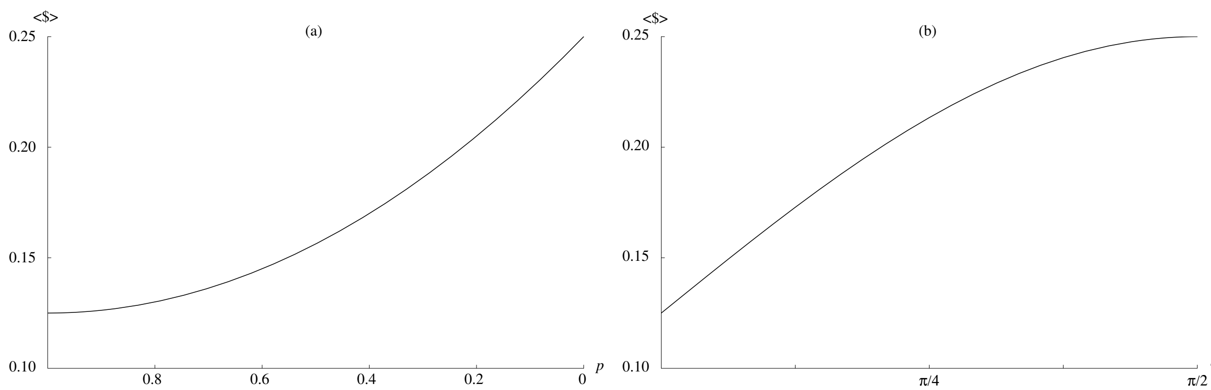

Figure 3. (a) The Nash equilibrium payoff in a four player quantum Minority game as a function of the decoherence probability p. The decoherence goes from the unperturbed quantum game at $p = 0$ (right) to maximum decoherence at $p = 1$ (left) where the classical result is reproduced. Compare this with (b) the Nash equilibrium payoff as a function of the entangling parameter γ.

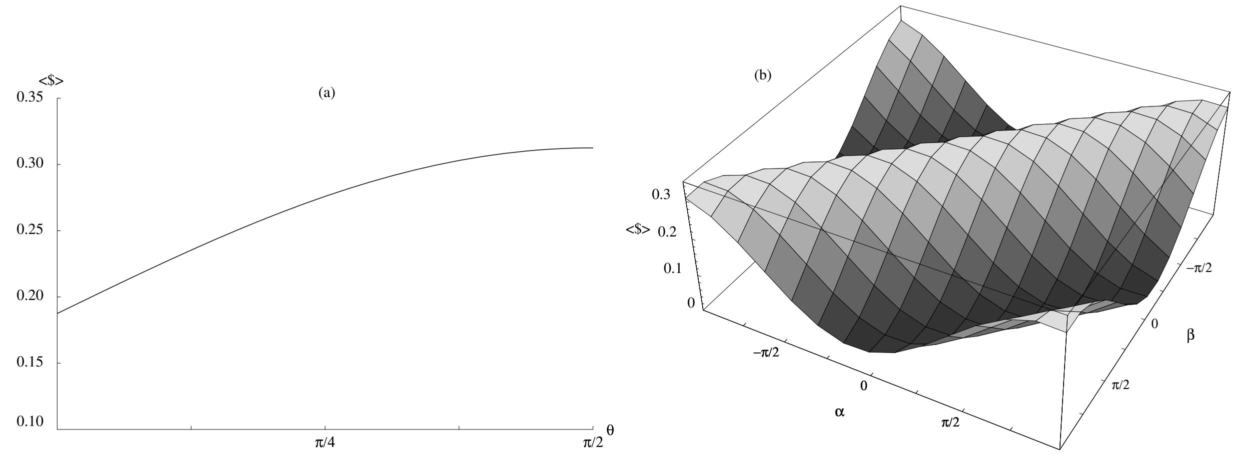

Figure 4. The payoff to the first player in a six player quantum Minority game when they choose the strategy (a) $\hat{U}(\theta, -\pi/24, \pi/24)$ or (b) $\hat{U}(\pi/2, \alpha, \beta)$, while the other players all select $\hat{s}_{\mathrm{NE}}$.

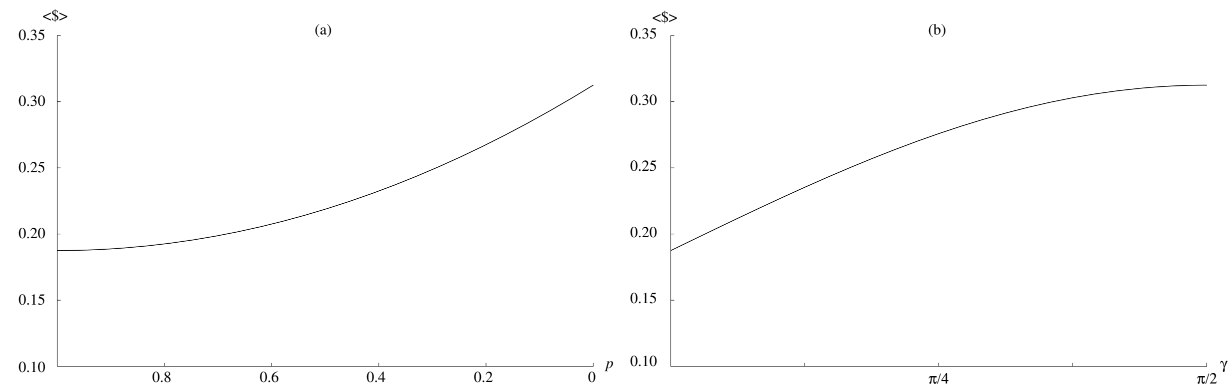

Figure 5. (a) The Nash equilibrium payoff in a six player quantum Minority game as a function of the decoherence probability p. The decoherence goes from the unperturbed quantum game at $p = 0$ (right) to maximum decoherence at $p = 1$ (left) where the classical result is reproduced. Compare this with (b) the Nash equilibrium payoff as a function of the entangling parameter γ.

of the balanced states is optimal since these are the ones for which no player scores. Each value of δ gives a NE for the N-even player quantum Minority game. In addition, for each δ there is a continuum of symmetric NE strategies of the form $\hat{U}(\pi/2, \eta - \delta, \eta + \delta)$. For $N > 4$ the payoffs for these strategies are sub-optimal. For example, for $N = 6$ each player scores $\frac{5}{16}$ compared with the maximum achievable payoff of $\frac{1}{3}$ that would result if all the final states consisted of two players selecting one option while the other four chose the second option. Figures 4 and 5 give some results for the six player quantum Minority game.

When N is odd the situation is changed. The collective optimal situation would be for $(N-1)/2$ players to select one alternative and the remainder to select the other. In this way the number of players that receive a reward is maximized. In the quantum game there is no way to achieve this with a symmetric strategy profile. Indeed, the quantum players in a one-shot quantum Minority game cannot improve upon the best classical result in the absence of cooperation, that achieved by selecting a random alternative.

4. CONCLUSION

We have considered a quantum version of an N-player Minority game where agents individually strive to select the minority alternative out of two possibilities. Entanglement amongst the qubits representing the players' selection offers the possibility of enhancing the payoffs to the players compared with the classical case. When the number of agents is odd, there is no quantum strategy that, when played by all parties, produces an expected payoff exceeding the best classical payoff. The latter can be simply achieved by selecting an alternative with an unbiased coin. However, when N is even, there exist multiple symmetric Nash equilibria obtained when each player selects a strategy from the set $\hat{U}(\pi/2, \eta - \delta, \eta + \delta)$ where $\delta = (4n + 1)\pi/(4N),\ n = 0, \pm 1, \pm 2, \dots$ and $\eta \in \{-\pi + \delta, \pi - \delta\}$. In this case the players achieve an expected payoff better than any that can be obtained in a non-cooperative classical game, but one that is the maximum possible for a symmetric strategy profile only when $N = 4$.

When decoherence is added to the quantum Minority game, the Nash equilibrium payoff is reduced as the decoherence is increased, as one would expect. However, the strategy profile remains a Nash equilibrium and is still the best result for the group that can be achieved in the absence of cooperation.

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